Implementable coupling of Lévy process and Brownian motion

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Problem formulation

Consider a Lévy process $X = (X(t), t \geqslant 0)$ with $\mathbb{E}X(1) = 0, \mathbb{E}X(1)^2 = 1$.

Problem

Construct a standard Brownian motion W on the same probability space (or its extension) such that:

- $\mathbb{E}\sup_{t\in[0,1]} |X(t) - W(t)|^2$ *is small,*
- *Brownian trajectories can be efficiently generated given a path of X.*

Comments:

- Other loss metrics can be used,
- Partial knowledge of X trajectory will be required.
- Comonotonic coupling of $X(1)$ and $W(1)$ produces minimal $\mathbb{E}|X(1) - W(1)|^2$.

Illustration I: drifted compound Poisson

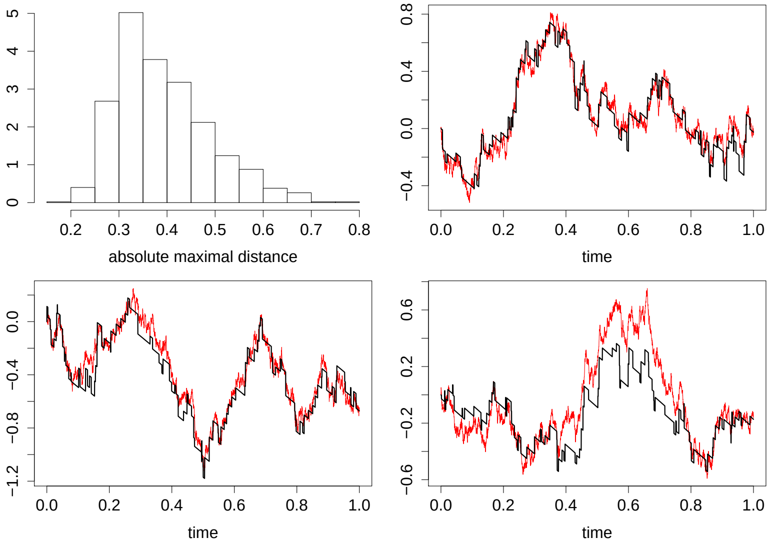


Figure: Three pairs of paths corresponding to $0.05, 0.50, 0.95$ quantiles.

Application I: stress testing and stochastic programming

Estimate a (tight) upper bound on the respective Wasserstein distance:

$$d^2_{\mathcal{W}}(X, W) \leqslant \mathbb{E} \sup_{t \in [0,1]} |X(t) - W(t)|^2.$$

Applications:

- Model risk [Blanchet & Murthy 19]: Brownian baseline model with a neighborhood containing a given Lévy model.
- Distributionally robust optimization [Esfahani & Kuhn 18]: allow for some freedom in the model to mitigate the optimizer's curse (links to regularization).

Application II: multilevel Monte Carlo

Estimate $\mathbb{E}g(X)$ by sampling an approximation $g(X_n)$ and control the bias. Standard approximation [Asmussen & Rosinski 01]: replace small jump martingale by a scaled Brownian motion.

MLMC [Giles 15]:

- sample $g(X_n)$ and $g(X_{n+1})$ jointly in a way that the level variance $\mathrm{Var}[g(X_{n+1}) - g(X_n)] \leqslant L^2 \cdot \mathbb{E}\sup_{t\in[0,1]}|X_{n+1}(t) - X_n(t)|^2$ is small,
- no significant increase in the cost,

Problem: couple the martingale of jumps in $[-\varepsilon_n, -\varepsilon_{n+1}) \cup (\varepsilon_{n+1}, \varepsilon_n]$ with a scaled Brownian motion.

Literature: existence of couplings

Lévy processes and random walks:

- [Skorokhod]: if $X_n(1) \xrightarrow{d} W(1)$ then there exists a coupling with $\sup_{t\in[0,1]}|X_n(t) - W(t)| \xrightarrow{\mathbb{P}} 0$.
- [Strassen 64]: random walk approximation by W underlying the functional LIL and based on Skorokhod's embedding.
- [Komlós, Major, Tusnády 75]: Hungarian embedding or the KMT coupling (based on conditional distributions).
- [Khoshnevisan 93]: construction of a drifted Poisson process X from W (more general construction induces dependence between inter-arrivals and subsequent jumps).
- ...

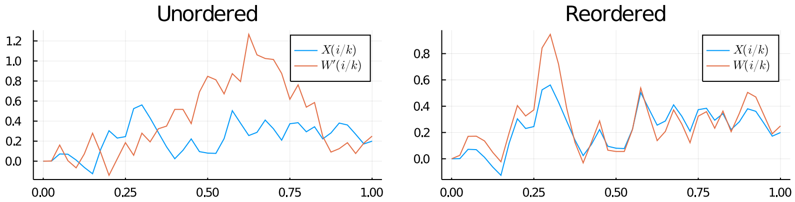
Main coupling: reordering of Brownian increments

Input:

- integer $k \geqslant 1$ (number of increments),
- k increments $\Delta_i^k X = X(i/k) - X((i-1)/k)$,
- the law of $X(1)$ or the ability to simulate from that.

Construction:

- Let $W'(1)$ be (nearly) comonotonically coupled with $X(1)$,
- Take an independent Brownian bridge $W'(t) - tW'(1), t \in [0, 1]$,
- Define W by reordering the k increments of W' according to the ordering of $\Delta_i^k X$.



Main coupling: details of construction

- Take k independent uniforms $U_1, \dots, U_k$ (for breaking ties),
- Let π be an a.s. unique random permutation on $\{1, \dots, k\}$ such that for all $i \neq j$:

$$\Delta^k_{\pi(i)} W' < \Delta^k_{\pi(j)} W' \quad \text{iff} \quad \Delta^k_i X < \Delta^k_j X \quad \text{or} \quad \Delta^k_i X = \Delta^k_j X, \ \ U_i < U_j.$$

- Define W by setting $W(0) := 0$ and

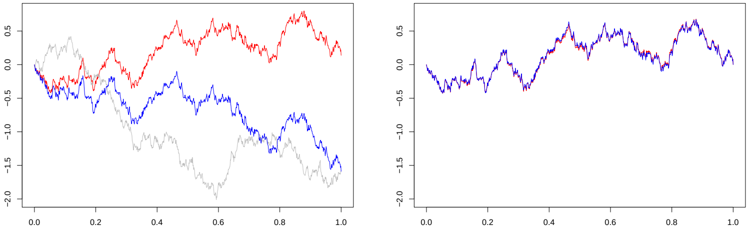
$$W(t) := W\left(\tfrac{i-1}{k}\right) + W'\left(\tfrac{\pi(i)-1}{k} + t - \tfrac{i-1}{k}\right) - W'\left(\tfrac{\pi(i)-1}{k}\right)$$

when $\frac{i-1}{k} < t \leqslant \frac{i}{k}$.

Hierarchical/recursive construction is possible!

On the choice of k

- $k = 1$: only the end-points are coupled, but the Brownian bridge is independent of X.
- $k = \infty$: the same if X has no Brownian component [González Cázares & Ivanovs 21] – this is a way to recover the Brownian part of a Lévy process.



Goal: asymptotic theory suggesting an adequate choice of k!

Illustration II

$$\Pi^0(\mathrm{d}x) = \left(0.4|x|^{-\alpha-1}1_{\{x\in(-\varepsilon_1,-\varepsilon_2)\}} + 0.6x^{-\alpha-1}1_{\{x\in(\varepsilon_2,\varepsilon_1)\}}\right)\mathrm{d}x, \quad \alpha = 1.5$$

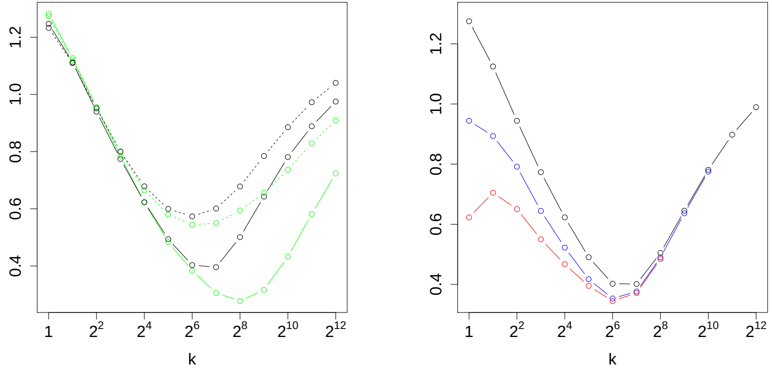


Figure: Root-mean-squared-maximal distances between X and W.
Left: four processes X. Right: second-level reordering with $k_2 \in \{1, 4, 16\}$.

Auxiliary method: comonotonic coupling of increments

Comonotonic coupling of $\zeta_1 = X(1/k)$ and $\zeta_2 = W(1/k)$ with cdf's F_i:

$$\zeta_2 = h(\zeta_1, U), \qquad h(x,u) = F_2^{-1}\big(\mathbb{P}(\zeta_1 < x) + u\mathbb{P}(\zeta_1 = x)\big).$$

Construction:

- Take Brownian increments $h(\Delta_i^k X, U_i)$ comonotonically coupled with ΔX_i^k; the same U_i used for ties.
- Accumulate these increments and use independent Brownian bridges to define $\widehat{W}(t)$.

Lemma

The processes W and $\widehat{W}$ are standard Brownian motions and their increments have the same ordering:

$$\Delta_i^k W < \Delta_j^k W \qquad \textit{iff} \qquad \Delta_i^k \widehat{W} < \Delta_j^k \widehat{W}$$

with probability 1.

Result: a trivariate process $(X, W, \widehat{W})$, and not just two couplings!
$\widehat{W}$ is less appealing, but its quality analysis is simpler.

Proximity of the two Brownian bridges

Lemma

It holds that

$$\mathbb{E} \max_{1 \leqslant i \leqslant k} \left| [W(\tfrac{i}{k}) - \tfrac{i}{k} W(1)] - [\widehat{W}(\tfrac{i}{k}) - \tfrac{i}{k} \widehat{W}(1)] \right|^2 = \mathrm{O}(\log \log k / k), \quad k \to \infty,$$

uniformly for all processes X.

Brownian discretization error $\mathrm{O}(\log k / k)$ yields:

$$\mathbb{E} \sup_{t \in [0,1]} \left| [W(t) - tW(1)] - [\widehat{W}(t) - t\widehat{W}(1)] \right|^2 = \mathrm{O}(\log k / k)$$

Asymptotic equivalence of the two coupling methods

- A sequence of Lévy processes $X_n \xrightarrow{d} W$,
- integers $k_n \to \infty$,
- coupled Brownian motions $W_n, \widehat{W}_n$.

Theorem

Assume

$$\mathbb{E} \sup_{t\in[0,1]} \left|X_n(t) - \widehat{W}_n(t)\right|^2 = \mathrm{O}(\varepsilon_n)$$

for some $\varepsilon_n \downarrow 0$. *Then also*

$$\mathbb{E} \sup_{t\in[0,1]} \left|X_n(t) - W_n(t)\right|^2 = \mathrm{O}(\varepsilon_n),$$

given it is true for $t = 1$ *and* $\log k_n / k_n = \mathrm{O}(\varepsilon_n)$.

Note: true when W and $\widehat{W}$ are swapped.

Asymptotic quality

Assumption:

$$\mathbb{E}X_n(1) = 0, \quad \mathbb{E}X_n^2(1) = 1, \quad \mathbb{E}X_n^4(1) < \infty.$$

Under mild conditions $X_n \xrightarrow{d} W$ implies $\mu_{4,n} := \int_{\mathbb{R}} x^4 \Pi_n(\mathrm{d}x) \to 0$, where Π_n is the Lévy measure of X_n.

Theorem

Under the above conditions we have

$$\mathbb{E} \sup_{t\in[0,1]} |X_n(t) - \widehat{W}_n(t)|^2 = \mathrm{O}(k_n\mu_{4,n} + \log k_n/k_n).$$

Corollary

Taking $k_n \sim \sqrt{|\log \mu_{4,n}|/\mu_{4,n}}$ *yields*

$$\mathbb{E} \sup_{t\in[0,1]} |X_n(t) - W_n(t)|^2 = \mathrm{O}\big(\log k_n/k_n\big) = \mathrm{O}\big(\sqrt{\mu_{4,n}|\log \mu_{4,n}|}\big).$$

Proof ingredients

By Doob's maximal inequality and comonotonic coupling bound:

$$\mathbb{E}\max_{i\leqslant k_n}\left|X_n(i/k_n)-\widehat{W}_n(i/k_n)\right|^2 \leqslant 4\mathbb{E}\left|X_n(1)-\widehat{W}_n(1)\right|^2$$
$$= 4k_n\mathbb{E}\left|X_n(1/k_n)-\widehat{W}_n(1/k_n)\right|^2 \leqslant 4Ck_n\mu_{4,n}.$$

Non-trivial discretization bound (for fixed n):

$$\mathbb{E}\sup_{t\in[0,1]}\left(X(t)-X^{[k]}(t)\right)^2 \leqslant C(k\mu_4+\log k/k).$$

Note: discretization error $\sup_{t\in[0,1]}\left(X(t)-X^{[k]}(t)\right)$ converges to the largest jump a.s.

Illustration III: the bounds are good!

- thresholds: $\varepsilon_{1,n} = 2^{-n}$ and $\varepsilon_{2,n} = 2^{-n-1}$.
- the optimal root-mean-squared-maximal distance d_n^* and k_n^*.

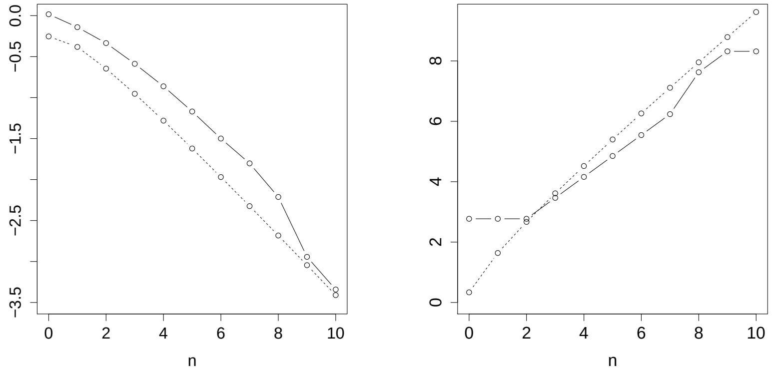


Figure: Left: $\log d_n^*$ (solid) and theoretical $\log(\mu_{4,n}|\log \mu_{4,n}|)/4$ (dashed). Right: $\log k_n^*$ (solid) and $\log(|\log \mu_{4,n}|/\mu_{4,n})/2$ (dashed).

Limiting regimes

- Zooming-out regime: $X_n(t) = X(nt)/\sqrt{n}$.

$$\mathbb{E} \sup_{t\in[0,1]} |X_n(t) - W_n(t)|^2 = \mathrm{O}\left(\sqrt{\log n/n}\right),$$

choosing $k_n \sim \sqrt{n \log n}$.

- Small jump Gaussian approximation: $\sigma_\varepsilon^2 := \int_{[-\varepsilon,\varepsilon]} x^2 \Pi(\mathrm{d}x)$, $\frac{\varepsilon}{\sigma_\varepsilon} \to 0$, and scaled $X_\varepsilon(t)$ to have variance t.

$$\mathbb{E} \sup_{t\in[0,1]} |X_\varepsilon(t) - W_\varepsilon(t)|^2 = \mathrm{O}\left(\frac{\varepsilon}{\sigma_\varepsilon}\sqrt{\left|\log\left(\frac{\varepsilon}{\sigma_\varepsilon}\right)\right|}\right),$$

choosing $k_\varepsilon \sim \sigma_\varepsilon \varepsilon^{-1}\sqrt{|\log \varepsilon|}$.

BG index $\beta \in (0, 2]$ and RV yields $\mathrm{O}(\varepsilon^{\beta_-/2})$, $\beta_- < \beta$.

- ...

Multilevel MC: mean complexity $\mathbb{E}\mathcal{C}_\delta$

There exists an MLMC algorithm with RMSE $\leqslant \delta$ s.t.

$$\mathbb{E}\mathcal{C}_\delta \overset{\log}{\sim} (1/\delta)^p, \qquad \delta \downarrow 0.$$

- Our coupling (intermediate jumps): $p = (5 - 4/\beta) \vee 2$,
- Standard independent sampling: $p = (6 - 4/\beta) \vee 2$,
- $\beta \in [0, 2]$ is the Blumenthal–Getoor index.

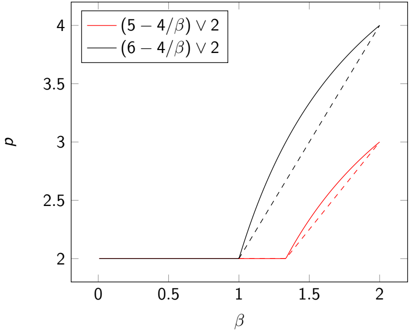


Figure: Dashed when $g(X_n)$ can be simulated exactly.

Conclusions

- Two asymptotically equivalent algorithms for construction of Brownian paths
  - reordering of Brownian increments,
  - comonotonic coupling of increments.
- asymptotic analysis as $X_n \stackrel{d}{\to} W$,
- adequate choice of k,
- implications for various limiting regimes,
- MLMC application: $\mathbb{E}\mathcal{C}_\delta \stackrel{\log}{\sim} (1/\delta)^{p_0-1}$.

Thank you!